

# Poincaré lemma and global homotopy formulas with sharp anisotropic Hölder estimates in $q$ -concave CR manifolds

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In this paper we prove sharp anisotropic Hölder estimates for the local solutions of the tangential Cauchy-Riemann equation in  $q$ -concave CR manifolds and we derive the same kind of estimates for global solutions when the manifold is compact.

It is wellknown since the works by Folland and Stein [4] that the sharp Hölder estimates for the solutions of the  $\bar{\partial}_b$ -equation have to be anisotropic, the complex tangential directions to  $M$  playing a special role. We consider two types of anisotropic Hölder spaces the spaces  $\mathcal{A}^{p+\alpha}$  and the spaces  $\Gamma^{p+\alpha}$ , which are defined respectively in Section 2 and in Section 3.

Our first result is a Poincaré Lemma for the  $\bar{\partial}_b$  operator :

**Theorem 0.1.** *Let  $M$  be a  $q$ -concave generic CR submanifold of class  $\mathcal{C}^3$  and of real codimension  $k$  of an  $n$ -dimensional complex manifold  $X$  and  $z_0$  a point in  $M$ . For any open neighborhood  $U$  of  $z_0$  in  $M$ , there exists an open neighborhood  $V \subset U$  of  $z_0$  and, for each  $r$  such that  $1 \leq r \leq q-1$  or  $n-k-q+1 \leq r \leq n-k$ , an operator*

$$T_r : \mathcal{C}_{n,r}(U) \rightarrow \mathcal{C}_{n,r}(V)$$

with the following properties:

- (i)  $f|_V = \bar{\partial}_b T_r f + T_{r+1} \bar{\partial}_b f$ , for  $1 \leq r \leq q-2$  or  $n-k-q+1 \leq r \leq n-k$ ;
- (ii)  $f|_V = \bar{\partial}_b T_r f$ , if  $r = q-1$  and  $\bar{\partial}_b f = 0$ ;
- (iii) if  $f \in \mathcal{A}_{n,r}^{p+\alpha}(U)$ ,  $0 < \alpha < 1$ , then  $T_r f \in \mathcal{A}_{n,r}^{p+1+\alpha}(V)$  if  $M$  is of class  $\mathcal{C}^{[\frac{p+1}{2}]+3}$  and  $1 \leq r \leq q-1$  or if  $M$  is of class  $\mathcal{C}^{[\frac{p}{2}]+3}$  and  $n-k-q+1 \leq r \leq n-k$ ;
- (iv) if  $f \in \Gamma_{n,r}^{p+\alpha}(U)$ ,  $0 < \alpha < 1$ , then  $T_r f \in \Gamma_{n,r}^{p+1+\alpha}(V)$  if  $M$  is of class  $\mathcal{C}^{p+4}$  and  $1 \leq r \leq q-1$  or if  $M$  is of class  $\mathcal{C}^{p+2}$  and  $n-k-q+1 \leq r \leq n-k$ .

The operators  $T_r$  are the operators defined in [2]. Theorem 0.1 is already proved in [2] (cf. Theorem 5.9) with  $\mathcal{C}^l$  estimates, it is derived from a local homotopy formula (cf. Proposition 1.1 of the present paper). The novelty here are the anisotropic Hölder regularity properties of the operators  $T_r$ , which are proved in Sections 2 and 3.

The second result is a global homotopy formula with sharp anisotropic Hölder estimates for compact CR manifolds.

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**Theorem 0.2.** *Let  $E$  be an holomorphic vector bundle over  $X$  and  $M$  be a compact,  $\mathcal{C}^\infty$ -smooth,  $q$ -concave generic CR submanifold of real codimension  $k$  of an  $n$ -dimensional complex manifold  $X$ . Assume the  $\bar{\partial}_b$ -cohomology group  $H^{n,r}(M)$  vanishes for some  $r$ ,  $0 \leq r \leq q-1$  or  $n-k-q+1 \leq r \leq n-k$ , then there exist continuous linear operators*

$$A_s : \mathcal{C}_{n,s}(M, E) \rightarrow \mathcal{C}_{n,s-1}(M, E), \quad s = r, r+1$$

such that:

(i) For all  $f \in \mathcal{C}_{n,r}(M, E)$  with  $\bar{\partial}_b f \in \mathcal{C}_{n,r+1}(M, E)$ ,

$$f = \begin{cases} A_1 \bar{\partial}_b f & \text{if } r = 0, \\ \bar{\partial}_b A_r f + A_{r+1} \bar{\partial}_b f & \text{if } r \geq 1 \end{cases} \quad (0.1)$$

(ii) For all  $p \in \mathbb{N}$  and  $0 < \alpha < 1$

$$A_s(\mathcal{A}_{n,s}^{p+\alpha}(M, E)) \subset \mathcal{A}_{n,s-1}^{p+1+\alpha}(M, E)$$

and  $A_s$  is continuous as an operator between  $\mathcal{A}_{n,s}^{p+\alpha}(M, E)$  and  $\mathcal{A}_{n,s-1}^{p+1+\alpha}(M, E)$ ;

(iii) For all  $p \in \mathbb{N}$  and  $0 < \alpha < 1$

$$A_s(\Gamma_{n,s}^{p+\alpha}(M, E)) \subset \Gamma_{n,s-1}^{p+1+\alpha}(M, E)$$

and  $A_s$  is continuous as an operator between  $\Gamma_{n,s}^{p+\alpha}(M, E)$  and  $\Gamma_{n,s-1}^{p+1+\alpha}(M, E)$ .

The operators  $A_r$  are derived from the local operators  $T_r$  by the globalization process from [7] and [3]. Then Theorem 0.2 follows immediately from Theorem 1.2 and the estimates in Sections 2 and 3.

The Poincaré Lemma for the  $\bar{\partial}_b$  operator with sharp anisotropic Hölder estimates is new in the case of CR manifolds of arbitrary codimension, it was proved first for the Heisenberg group and more generally for strictly pseudoconvex hypersurfaces by Folland and Stein [4]. A related result in CR manifolds of arbitrary codimension is a local almost homotopy formula with sharp anisotropic Hölder estimates proved by Polyakov in [9], but there is a mistake in the construction of the barriers which is corrected in [10] and we hope that the same anisotropic estimates hold for his new kernels.

The global result is not totally new, it is proved in [11] even for abstract CR manifolds for the anisotropic Sobolev spaces (Sobolev version of the  $\mathcal{A}^{p+\alpha}$  spaces) and for the Folland-Stein spaces  $\Gamma^{p+\alpha}$  if  $r \neq q-1$ . The proof is based on  $L^2$  theory for the  $\square_b$  operator and the Hodge decomposition theorem which gives a global homotopy formula in degree  $r$  but only if the CR manifold  $M$  is  $(r+2)$ -concave.

Our result contains the important case where  $M$  is only 2-concave and  $r = 1$ , which may be useful in the study of the embeddability of compact CR manifolds (in [11], results in degree 1 need the manifold to be 3-concave).

## 1 Preliminaries and definitions

Let  $(\mathbb{M}, H_{0,1}\mathbb{M})$  be a generically embeddable abstract compact CR manifold of class  $\mathcal{C}^\infty$  and  $\mathcal{E} : \mathbb{M} \rightarrow M \subset X$  be a  $\mathcal{C}^\infty$ -smooth CR generic embedding of  $\mathbb{M}$  in a complex

manifold  $X$ , then  $M$  is a compact CR submanifold of  $X$  of class  $\mathcal{C}^\infty$  with the CR structure  $H_{0,1}M = d\mathcal{E}(H_{0,1}\mathbb{M}) = T_{\mathbb{C}}M \cap T_{0,1}X$  and the tangential Cauchy-Riemann operator  $\bar{\partial}_b$  induced by the Cauchy-Riemann operator  $\bar{\partial}$  from the complex manifold  $X$ .

An generic CR manifold  $\mathbb{M}$  is said to be  $q$ -concave if its Levi form at each point admits at least  $q$  negative eigenvalues in all directions.

Assume  $\mathbb{M}$  is  $q$ -concave, then  $M$  is also  $q$ -concave and we may apply the results in [2] and [7], [3] on local estimates and global homotopy formulas for the tangential Cauchy-Riemann operator.

First let us recall the definition of the usual Hölder spaces of forms. If  $D$  is relatively compact domain in  $X$ , then

-  $\mathcal{C}^\alpha(\bar{D} \cap M)$ ,  $0 \leq \alpha < 1$ , is the set of continuous functions on  $\bar{D} \cap M$  which are Hölder continuous with exponent  $\alpha$  on  $\bar{D} \cap M$ , if  $\alpha > 0$ . We set

$$\|f\|_\alpha = \sup_{z \in D \cap M} |f(z)| + \sup_{\substack{z, \zeta \in D \cap M \\ z \neq \zeta}} \frac{|f(z) - f(\zeta)|}{|z - \zeta|^\alpha} \quad (1.1)$$

-  $\mathcal{C}^{l+\alpha}(\bar{D} \cap M)$ ,  $l \in \mathbb{N}$ ,  $0 \leq \alpha < 1$ , is the space of functions of class  $\mathcal{C}^l$  on  $\bar{D} \cap M$ , whose derivatives of order  $l$  are in  $\mathcal{C}^\alpha(\bar{D} \cap M)$ .

The Hölder space  $\mathcal{C}_*^{l+\alpha}(\bar{D} \cap M)$ ,  $l \in \mathbb{N}$ ,  $0 \leq \alpha < 1$ , is then the space of continuous forms on  $\bar{D} \cap M$ , whose coefficients are in  $\mathcal{C}^{l+\alpha}(\bar{D} \cap M)$ .

In [2] the following result is proved

**Proposition 1.1.** *Let  $M$  be a  $q$ -concave generic CR submanifold of  $X$  of class  $\mathcal{C}^\infty$ . For each point in  $M$ , there exist a neighborhood  $U$  and linear operators*

$$T_r : \mathcal{C}_{n,r}^0(M) \rightarrow \mathcal{C}_{n,r-1}^0(U), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k,$$

with the following two properties :

(i) For all  $l \in \mathbb{N}$  and  $1 \leq r \leq q$  or  $n - k - q + 1 \leq r \leq n - k$ ,

$$T_r(\mathcal{C}_{n,r}^l(M)) \subset \mathcal{C}_{n,r-1}^{l+1/2}(\bar{U})$$

and  $T_r$  is continuous as an operator between  $\mathcal{C}_{n,r}^l(M)$  and  $\mathcal{C}_{n,r-1}^{l+1/2}(\bar{U})$ .

(ii) If  $f \in \mathcal{C}_{n,r}^1(M)$ ,  $0 \leq r \leq q - 1$  or  $n - k - q + 1 \leq r \leq n - k$ , has compact support in  $U$ , then on  $U$ ,

$$f = \begin{cases} T_1 \bar{\partial}_b f & \text{if } r = 0, \\ \bar{\partial}_b T_r f + T_{r+1} \bar{\partial}_b f & \text{if } 1 \leq r \leq q - 1 \text{ or } n - k - q + 1 \leq r \leq n - k. \end{cases} \quad (1.2)$$

and in [7] and [3] we have derive from the previous proposition a global homotopy formula by mean of a functional analytic construction:

**Theorem 1.2.** *Let  $E$  be an holomorphic vector bundle over  $X$  and  $M$  be a compact  $q$ -concave generic CR submanifold of  $X$  of class  $\mathcal{C}^\infty$ . Then there exist finite dimensional subspaces  $\mathcal{H}_r$  of  $\mathcal{Z}_{n,r}^\infty(M, E)$ ,  $0 \leq r \leq q - 1$  and  $n - k - q + 1 \leq r \leq n - k$ , where  $\mathcal{H}_0 = \mathcal{Z}_{n,0}^\infty(M, E)$ , continuous linear operators*

$$A_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r-1}^0(M, E), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k$$

and continuous linear projections

$$P_r : \mathcal{C}_{n,r}^0(M, E) \rightarrow \mathcal{C}_{n,r}^0(M, E), \quad 0 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k,$$

with

$$\text{Im } P_r = \mathcal{H}_r, \quad 0 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k, \quad (1.3)$$

and

$$\mathcal{C}_{n,r}^0(M, E) \cap \bar{\partial}_b \mathcal{C}_{n,r-1}^0(M, E) \subseteq \text{Ker } P_r, \quad 1 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k, \quad (1.4)$$

such that:

(i) For all  $l \in \mathbb{N}$  and  $1 \leq r \leq q$  or  $n-k-q+1 \leq r \leq n-k$ ,

$$A_r(\mathcal{C}_{n,r}^l(M, E)) \subset \mathcal{C}_{n,r-1}^{l+1/2}(M, E)$$

and  $A_r$  is continuous as an operator between  $\mathcal{C}_{n,r}^l(M, E)$  and  $\mathcal{C}_{n,r-1}^{l+1/2}(M, E)$ .

(ii) For all  $0 \leq r \leq q-1$  or  $n-k-q+1 \leq r \leq n-k$  and  $f \in \mathcal{C}_{n,r}^0(M, E)$  with  $\bar{\partial}_b f \in \mathcal{C}_{n,r+1}^0(M, E)$ ,

$$f - P_r f = \begin{cases} A_1 \bar{\partial}_b f & \text{if } r = 0, \\ \bar{\partial}_b A_r f + A_{r+1} \bar{\partial}_b f & \text{if } 1 \leq r \leq q-1 \text{ or } n-k-q+1 \leq r \leq n-k. \end{cases} \quad (1.5)$$

We have now to recall the main steps of the construction of the local kernels defining the operators  $T_r$ .

Let  $M$  be a generic  $CR$  manifold of class  $\mathcal{C}^3$  in  $\mathbb{C}^n$ ,  $z_0$  a point in  $M$  and  $U_0$  an open neighborhood of  $z_0$  in  $\mathbb{C}^n$  and  $\hat{\rho}_1, \dots, \hat{\rho}_k$  some functions of class  $\mathcal{C}^3$  from  $U_0$  into  $\mathbb{R}$  such that

$$M \cap U_0 = \{z \in U_0 \mid \hat{\rho}_1(z) = \dots = \hat{\rho}_k(z) = 0\}$$

and satisfying  $\bar{\partial} \hat{\rho}_1(z) \wedge \dots \wedge \bar{\partial} \hat{\rho}_k(z) \neq 0$  for  $z \in M \cap U_0$ .

Let  $C > 0$  be a fixed constant, we set, for  $j = 1, \dots, k$ ,

$$\begin{aligned} \rho_j &= \hat{\rho}_j + C \sum_{\nu=1}^k \hat{\rho}_\nu^2 \\ \rho_{-j} &= -\hat{\rho}_j + C \sum_{\nu=1}^k \hat{\rho}_\nu^2. \end{aligned} \quad (1.6)$$

We define  $\mathcal{I}$  as the set of all subsets  $I \subset \{\pm 1, \dots, \pm k\}$  such that  $|i| \neq |j|$  for all  $i, j \in I$  with  $i \neq j$ . For  $I \in \mathcal{I}$ ,  $|I|$  denotes the number of elements in  $I$ , then  $\mathcal{I}(l)$ ,  $1 \leq l \leq k$ , is the set of all  $I \in \mathcal{I}$  with  $|I| = l$  and  $\mathcal{I}'(l)$ ,  $1 \leq l \leq k$ , is the set of all  $I \in \mathcal{I}$  of the form  $I = (i_1, \dots, i_l)$  with  $|i_\nu| = \nu$  for  $\nu = 1, \dots, l$ .

If  $I \in \mathcal{I}$  and  $\nu \in \{1, \dots, |I|\}$ , then  $i_\nu$  is the element with rank  $\nu$  in  $I$  after ordering  $I$  by modulus. We set  $I(\hat{\nu}) = I \setminus \{i_\nu\}$ .

If  $I \in \mathcal{I}$ , then

$\text{sgn} I = 1$  if the number of negative elements in  $I$  is even  
 $\text{sgn} I = -1$  if the number of negative elements in  $I$  is odd.

Let  $(e_1, \dots, e_k)$  be the canonical basis of  $\mathbb{R}^k$ , set  $e_{-j} = -e_j$  for every  $1 \leq j \leq k$ . Let  $I = (i_1, \dots, i_l)$  be in  $\mathcal{I}(l)$ ,  $1 \leq l \leq k$ , set

$$\tilde{\Delta}_I = \{x = \sum_{j=1}^l \lambda_j e_{i_j} \mid \lambda_i \geq 0, 1 \leq i \leq l, \sum_{i=1}^l \lambda_i = 1\}.$$

We identify the abstract simplex  $\Delta_I$  with the geometric simplex  $\tilde{\Delta}_I$  by setting  $x(\lambda) = \sum_{j=1}^l \lambda_j e_{i_j}$  for all  $\lambda \in \Delta_I$

For all  $I \in \mathcal{I}'(k)$ , we denote by  $I_*$  the multi-index  $(i_1, \dots, i_k, *)$ , where  $I = (i_1, \dots, i_k)$ , and by  $\mathcal{I}'(k, *)$  the set of all multi-indexes  $I_*$ , with  $I \in \mathcal{I}'(k)$ . We set  $\rho_* = \frac{1}{k}(\rho_1 + \dots + \rho_k)$  and  $\rho_\lambda = \lambda_1 \rho_{i_1} + \dots + \lambda_k \rho_{i_k} + \lambda_* \rho_*$  for  $\lambda = (\lambda_1, \dots, \lambda_k, \lambda_*) \in \Delta_{I_*}$ .

We denote by  $D$  a relatively compact open subset of  $U_0$  and for  $I \in \mathcal{I}$ ,  $I = (i_1, \dots, i_{|I|})$ , we define

$$\begin{aligned} D_I &= \{\rho_{i_1} < 0\} \cap \dots \cap \{\rho_{i_{|I|}} < 0\} \cap D \\ D_I^* &= \{\rho_{i_1} > 0\} \cap \dots \cap \{\rho_{i_{|I|}} > 0\} \cap D \\ S_I &= \{\rho_{i_1} = 0, \dots, \rho_{i_{|I|}} = 0\} \cap D \\ \Gamma_I &= \{\rho_{i_1} = \dots = \rho_{i_{|I|}}\} \cap D_I \\ \Gamma_I^* &= \{\rho_{i_1} = \dots = \rho_{i_{|I|}}\} \cap D_I^*. \end{aligned}$$

These manifolds are oriented as follows :  $D_I$  and  $D_I^*$  as  $\mathbb{C}^n$  for all  $I \in \mathcal{I}$ ,  $S_{\{j\}}$  as the boundary of  $D_{\{j\}}$  for  $j = \pm 1, \dots, \pm k$ ,  $S_I$  as the boundary of  $S_{I(\widehat{|I|})} \cap \overline{D}_{\{i_{|I|}\}}$  for all  $I \in \mathcal{I}$ ,  $|I| \geq 2$ ,  $\Gamma_I$  such that  $S_I = \partial \Gamma_I$  and  $M \cap D$  as  $S_I$  with  $I = \{1, \dots, k\}$ .

If  $M$  is  $q$ -concave, it follows from Lemma 3.1.1 in [1] that we can choose the constant  $C$  in (1.6) such that the functions  $\rho_j$ ,  $-k \leq j \leq k$ ,  $j \neq 0$ , have the following property: for each  $I \in \mathcal{I}'(k)$  and every  $\lambda \in \Delta_I$ , the Levi form of the defining function  $\rho_\lambda$  of  $M$  in the direction  $x(\lambda)$  has at least  $q + k$  positive eigenvalues on  $U' \subset \subset U_0$ . Then using the method developed in section 3 of [5] and the results in [6], we can construct, for each  $\lambda \in \Delta_I$  a Leray section in the direction  $x(\lambda)$ , which has some holomorphy properties and depends smoothly on  $\lambda$  and we get on  $U' \times U' \setminus \Delta(U')$ ,  $U' \subset \subset U_0$ , some Cauchy-Fantappié kernels

$$C_{I*}(z, \zeta) = \int_{\lambda \in \Delta_{I*}} K_{I*}(z, \zeta, \lambda)$$

for each  $I \in \mathcal{I}'(k)$  (cf.[2]) such that, if we set  $R_M = \sum_{I \in \mathcal{I}'(k)} \text{sgn}(I) C_{I*}$ , the associated integral operators  $T_r$  satisfy the homotopy formula (ii) of Proposition 1.1 with  $U = D \cap M$ .

We can describe the singularity of the kernels  $C_{I*}$  in the following way.

A form of type  $O_s$  (or of type  $O_s(z, \zeta, \lambda)$ ) on  $\overline{D}_I \times \overline{D}_I^* \times \Delta_{I*}$  is, by definition, a continuous differential form  $f(z, \zeta, \lambda)$  defined for all  $(z, \zeta, \lambda) \in \overline{D}_I^* \times \overline{D}_I \times \Delta_{I*}$  with  $z \neq \zeta$  such that the following conditions are fulfilled :

1. All derivatives of the coefficients of  $f$  which are of order 0 in  $z$ , and of order  $\leq 1$  in  $\zeta$  and of arbitrary order in  $\lambda$  are continuous for all  $(z, \zeta, \lambda) \in \overline{D}_I^* \times \overline{D}_I \times \Delta_{I*}$  with  $z \neq \zeta$ .

2. Let  $\nabla_\zeta^\kappa$ ,  $\kappa = 0, 1$ , be a differential operator with constant coefficients, which is of order 0 in  $z$ , of order  $\kappa$  in  $\zeta$  and of arbitrary order in  $\lambda$ . Then there is a constant  $C > 0$  such that, for each coefficient  $\varphi(z, \zeta, \lambda)$  of the form  $f(z, \zeta, \lambda)$ ,

$$|\nabla_\zeta^\kappa \varphi(z, \zeta, \lambda)| \leq C |\zeta - z|^{s-\kappa}$$

for all  $(z, \zeta, \lambda) \in \overline{D}_I \times \overline{D}_I^* \times \Delta_{I*}$  with  $z \neq \zeta$ .

Following the calculations in [2], we get

$$[K_{I*}(z, \zeta, \lambda)]_{\deg \lambda = |I|} \wedge dz_1 \wedge \cdots \wedge dz_n = \sum_{\substack{0 \leq m \leq k \\ i_1, \dots, i_m \in I}} \frac{O_{|I|+1-m}}{\Phi^n} \wedge \partial \rho_{i_1}(z) \wedge \cdots \wedge \partial \rho_{i_m}(z). \quad (1.7)$$

and

$$[\overline{\partial}_z K_{I*}(z, \zeta, \lambda)]_{\deg \lambda = |I|} \wedge dz_1 \wedge \cdots \wedge dz_n = \sum_{\substack{0 \leq m \leq k \\ i_1, \dots, i_m \in I}} \frac{O_{|I|-m}}{\Phi^n} \wedge \partial \rho_{i_1}(z) \wedge \cdots \wedge \partial \rho_{i_m}(z). \quad (1.8)$$

The support function associated to the Leray section used to construct the kernels  $C_{I*}$  satisfies for  $\zeta, z$  in a neighborhood of  $U'$  and  $\lambda \in \Delta_{I*}$

$$\Phi(z, \zeta, \lambda) = 2 \sum_{j=1}^k \frac{\partial \rho_\lambda}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) + O_2 \quad (1.9)$$

and

$$\operatorname{Re} \Phi(z, \zeta, \lambda) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \gamma |\zeta - z|^2. \quad (1.10)$$

Consequently, if  $X_{\mathbb{C}}$  denotes a complex tangent vector field to  $M$ , then

$$X_{\mathbb{C}}^\zeta \Phi(z, \zeta, \lambda) = O_1 \quad \text{and} \quad X_{\mathbb{C}}^z \Phi(z, \zeta, \lambda) = O_1. \quad (1.11)$$

Moreover the function  $\Phi(z, \zeta, \lambda)$  is of class  $\mathcal{C}^\infty$  in its first variable  $z$  and of class  $\mathcal{C}^{l-1}$  in its second variable  $\zeta$  if the manifold  $M$  is of class  $\mathcal{C}^l$ .

In the next sections we shall prove that in fact the operators  $T_r$  and  $A_r$ ,  $1 \leq r \leq q$ , satisfy sharp anisotropic estimates.

## 2 First anisotropic estimates

If  $M$  is a generic CR submanifold of a complex manifold  $X$ , the tangent bundle to  $M$  admits a maximal complex subbundle called the complex tangent bundle to  $M$  and denoted by  $HM$ . It is related to the CR structure of  $M$  by  $HM = TM \cap (H_{0,1}M + \overline{H}_{0,1}M)$ .

Let us define now some anisotropic Hölder spaces of forms which specify the complex tangent directions:

-  $\mathcal{A}^\alpha(\overline{D} \cap M)$ ,  $0 < \alpha < 1$ , is the set of continuous functions on  $\overline{D} \cap M$  which are in  $\mathcal{C}^{\alpha/2}(\overline{D} \cap M)$ .

-  $\mathcal{A}^{1+\alpha}(\overline{D} \cap M)$ ,  $0 < \alpha < 1$ , is the set of functions  $f$  such that  $f \in \mathcal{C}^{(1+\alpha)/2}(\overline{D} \cap M)$  and  $X_{\mathbb{C}}f \in \mathcal{C}^{\alpha/2}(\overline{D} \cap M)$ , for all complex tangent vector fields  $X_{\mathbb{C}}$  to  $M$ . Set

$$\|f\|_{A\alpha} = \|f\|_{(1+\alpha)/2} + \sup_{\|X_{\mathbb{C}}\| \leq 1} \|X_{\mathbb{C}}f\|_{\alpha/2} \quad (2.1)$$

-  $\mathcal{A}^{p+\alpha}(\overline{D} \cap M)$ ,  $p \geq 2$ ,  $0 < \alpha < 1$ , is the set of functions  $f$  of class  $\mathcal{C}^{[p/2]}$  such that  $Xf \in \mathcal{A}^{p-2+\alpha}(\overline{D} \cap M)$ , for all tangent vector fields  $X$  to  $M$  and  $X_{\mathbb{C}}f \in \mathcal{A}^{p-1+\alpha}(\overline{D} \cap M)$ , for all complex tangent vector fields  $X_{\mathbb{C}}$  to  $M$ .

The anisotropic Hölder space of forms  $\mathcal{A}_*^{p+\alpha}(\overline{D} \cap M)$ ,  $p \geq 0$ ,  $0 < \alpha < 1$ , is then the space of continuous forms on  $\overline{D} \cap M$ , whose coefficients are in  $\mathcal{A}^{p+\alpha}(\overline{D} \cap M)$ .

This section is devoted to the study of the continuity properties with respect to these spaces of the operators  $T_r$  and  $A_r$ ,  $1 \leq r \leq q$  and  $n - k - q + 1 \leq r \leq n - k$ , defined in the previous section when  $M$  is  $q$ -concave.

We first prove some estimates for the operators defined by the local kernels from the previous section. For this we may assume that  $M$  is embedded in  $\mathbb{C}^n$ . To clarify the exposition, we introduce a new kind of operators.

**Definition 2.1.** Let  $m \in \mathbb{N}$ ,  $\epsilon \in \{-1, 0, 1\}$ ,  $\delta > 0$  and  $I \in \mathcal{I}'(k)$ . An operator of type  $(m, \epsilon, \delta)$  is, by definition, a map

$$E : \mathcal{C}_{n,*}^0(\Gamma_I) \rightarrow \mathcal{C}_{n,*}^\infty(\Gamma_I)$$

such that there exist

- an integer  $\kappa \geq 0$
- a differential form  $\widehat{E}(z, \zeta, \lambda)$  of type  $O_{|I|+1-2n+2\kappa+m+\epsilon}$  on  $\Gamma_I \times D_I^* \times \Delta_{I*}$  such that for all  $f \in \mathcal{C}_{n,*}^0(\Gamma_I)$

$$Ef(\zeta) = \int_{(z,\lambda) \in \Gamma_I \times \Delta_{I*}} \widetilde{f}(z) \wedge \frac{\widehat{E}(z, \zeta, \lambda) \wedge \Theta(z)}{(\Phi + \delta)^{\kappa+m}(z, \zeta, \lambda)},$$

where  $\widetilde{f} \in \mathcal{C}_{0,*}^0(\Gamma_I)$  is the form with

$$f(z) = \widetilde{f}(z) \wedge dz_1 \wedge \cdots \wedge dz_n,$$

and  $\Theta = 1$ , if  $m = 0$ , and either  $\Theta = \partial\rho_{i_1} \wedge \cdots \wedge \partial\rho_{i_m}$  or  $\Theta = \overline{\partial}\rho_{i_1} \wedge \cdots \wedge \partial\rho_{i_m}$ , if  $m \geq 1$  with  $i_1, \dots, i_m \in I$ .

We consider the operators  $\widetilde{C}_{I*}$ ,  $I \in \mathcal{I}'(k)$ , defined by

$$\widetilde{C}_{I*}f(\zeta) = \int_{z \in D \cap M} f(z) \wedge C_{I*}(z, \zeta) = \int_{(z,\lambda) \in D \cap M \times \Delta_{I*}} f(z) \wedge K_{I*}(z, \zeta, \lambda) \quad (2.2)$$

for  $f \in \mathcal{C}_{n,r}^1(D)$ ,  $0 \leq r \leq n - k$ . Using Stokes formula we get

$$\widetilde{C}_{I*}f(\zeta) = \int_{(z,\lambda) \in \Gamma_I \times \Delta_{I*}} \overline{\partial}f(z) \wedge K_{I*}(z, \zeta, \lambda) + (-1)^{n+r} \int_{(z,\lambda) \in \Gamma_I \times \Delta_{I*}} f(z) \wedge \overline{\partial}_z K_{I*}(z, \zeta, \lambda). \quad (2.3)$$

By (1.7) and ((1.8) the kernel  $\tilde{C}_{I*}$  is the sum of an operator of type  $(m, 1, 0)$  and of an operator of type  $(m, 0, 0)$ ,  $0 \leq m \leq k$ , and if  $X_{\mathbb{C}}$  denotes a complex tangent vector field to  $M$ , then by (1.11) the kernels  $X_{\mathbb{C}}^{\zeta} \tilde{C}_{I*}$  and  $X_{\mathbb{C}}^z \tilde{C}_{I*}$  are the sum of an operator of type  $(m, 0, 0)$  and of an operator of type  $(m, -1, 0)$ ,  $0 \leq m \leq k$ .

To prove the anisotropic estimates we need the following spaces and norms on differential forms :

-  $\mathcal{B}_*^{\beta}(\Gamma_I)$ ,  $\beta \geq 0$ , is the space of forms  $f \in \mathcal{C}_*^0(\overline{\Gamma}_I \setminus M)$  such that, for some constant  $C > 0$ ,

$$\|f(z)\| \leq C[\text{dist}(z, M)]^{-\beta}, \quad z \in \Gamma_I,$$

where  $\text{dist}(z, M)$  is the Euclidean distance between  $z$  and  $M$ . Set

$$\|f\|_{\beta} = \sup_{z \in \Gamma_I} (\|f(z)\| [\text{dist}(z, M)]^{\beta}) \quad (2.4)$$

for  $\beta \geq 0$  and  $f \in \mathcal{B}_*^{\beta}(\Gamma_I)$ .

It follows from the characterization of Hölder continuous functions by mean of the Poisson integral that, if  $f \in \mathcal{C}_*^{\alpha}(\overline{D} \cap M)$ ,  $0 < \alpha < 1$ , there exists for each  $I \in \mathcal{I}'(k)$  a continuous form  $f_I$  on  $\overline{\Gamma}_I$  such that  $f_{I|M} = f$  and  $X_I f_I \in \mathcal{B}_*^{1-\alpha}(\Gamma_I)$ , for all vector fields  $X_I$  tangent to  $\Gamma_I$ .

**Lemma 2.2.** *If  $f \in \mathcal{B}_*^{\beta}(\Gamma_I)$ ,  $0 \leq \beta < 1$ , has compact support in  $D$  and if  $E_I$  is an operator of type  $(m, \epsilon, \delta)$ , then there exists a constant  $C > 0$ , independent of  $\delta$ , such that for  $\zeta \in D_I^*$*

$$\begin{aligned} |E_I f(\zeta)| &\leq C \|f\|_{\beta} [\text{dist}(\zeta, M) + \delta]^{1/2-\beta}, \quad \text{if } \epsilon = 0 \\ |E_I f(\zeta)| &\leq C \|f\|_{\beta} [\text{dist}(\zeta, M) + \delta]^{-\beta}, \quad \text{if } \epsilon = -1 \\ |E_I f(\zeta)| &\leq C \|f\|_{\beta} [\text{dist}(\zeta, M) + \delta]^{1-\beta}, \quad \text{if } \epsilon = 1 \end{aligned}$$

*Proof.* It follows from [5] and [2] that, after integration in  $\lambda$ , an operator of type  $(m, \epsilon, \delta)$  is controled, since  $n \geq 3$ , by

$$\int_{\Gamma_I} \frac{|f_I| |\sigma \wedge d\rho_I \wedge_{\nu=1}^s dt_{\nu}|}{(|\rho_I| + d + \delta + |\zeta - z|^2) \Pi_{\nu=1}^s (|t_{\nu}| + d + \delta + |\zeta - z|^2)^{1+1/s} |\zeta - z|^{2n-k-s-3-\epsilon}},$$

with  $1 \leq s \leq k$ , where  $\rho_I$  is the function defined by  $\rho_I(z) = \rho_{i_1}(z) = \dots = \rho_{i_k}(z)$  for  $z \in \Gamma_I$ ,  $d = d(\zeta) = \text{dist}(\zeta, M)$ ,  $\sigma$  a monomial in  $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ ,  $t_{\nu} = \text{Im } \Phi(z, \zeta, \lambda^{\nu})$  and  $dt_{\nu} = d_z \text{Im } \Phi(z, \zeta, \lambda^{\nu})$  with  $\lambda^1, \dots, \lambda^{k+1}$  some points in  $\Delta_{I*}$  which are linearly independent in  $\mathbb{R}^{k+1}$ ,

We denote all the constants by  $C$ . Then since  $d(z) \leq C|\rho_I(z)|$  for  $z \in \Gamma_I$ , we get

$$\begin{aligned} \|E_I f(\zeta)\| &\leq \\ C \|f\|_{\beta} &\int_{\Gamma_I} \frac{|\rho_I|^{-\beta} |\sigma \wedge d\rho_I \wedge_{\nu=1}^s dt_{\nu}|}{(|\rho_I| + d + \delta + |\zeta - z|^2) \Pi_{\nu=1}^s (|t_{\nu}| + d + \delta + |\zeta - z|^2)^{1+1/s} |\zeta - z|^{2n-k-s-3-\epsilon}} \end{aligned}$$

for all  $f \in \mathcal{B}_*^{\beta}(\Gamma_I)$  and  $\zeta \in D_I^*$ . We can use  $\rho_I$  as a coordinate in  $\Gamma_I$  and since  $M$  is generic the  $t_{\nu}$ 's can also be used as coordinates, so we obtain that

$$\begin{aligned} \|E_I f(\zeta)\| &\leq \\ C \|f\|_{\beta} &\int_{\substack{y \in \mathbb{R}^{2n-k+1} \\ |y| < c}} \frac{|y_1|^{-\beta} dy_1 \wedge dy_2 \wedge \dots \wedge dy_{2n-k+1}}{(|y_1| + d + \delta + |\zeta - z|^2) \Pi_{\nu=1}^s (|y_{\nu}| + d + \delta + |y|^2)^{1+1/s} |y|^{2n-k-s-3-\epsilon}}. \end{aligned}$$



Since  $\beta < 1$ , we can integrate with respect to  $y_1$ , which gives

$$\|E_I f(\zeta)\| \leq C \|f\|_\beta \int_{\substack{y \in \mathbb{R}^{2n-k} \\ |y| < c}} \frac{dy_2 \wedge \cdots \wedge dy_{2n-k+1}}{(d + \delta + |y|^2)^\beta \prod_{\nu=1}^s (|y_\nu| + d + \delta + |y|^2)^{1+1/s} |y|^{2n-k-s-3-\epsilon}}.$$

Then integrating with respect to  $y_2, \dots, y_s$  and using spherical coordinates, we have

$$\|E_I f(\zeta)\| \leq C \|f\|_\beta \int_0^C \frac{dr}{(d + \delta + r^2)^{\beta+1} r^{-2-\epsilon}},$$

which proves the lemma following the values of  $\epsilon$ .  $\square$

In the same way, following again [5], we can prove estimates for the gradient of  $E_I f$ , when  $E_I$  is an operator of type  $(m, \epsilon, \delta)$  and  $f \in \mathcal{B}_*^\beta(\Gamma_I)$ ,  $0 \leq \beta < 1$ .

**Lemma 2.3.** *If  $f \in \mathcal{B}_*^\beta(\Gamma_I)$ ,  $0 \leq \beta < 1$ , has compact support in  $D$ ,  $E_I$  is an operator of type  $(m, \epsilon, \delta)$  and if  $\nabla_\zeta$  is one of the operators  $\frac{\partial}{\partial \zeta_1}, \dots, \frac{\partial}{\partial \zeta_n}$  or  $\frac{\partial}{\partial \zeta_1}, \dots, \frac{\partial}{\partial \zeta_n}$ , then there exists a constant  $C > 0$  independent of  $\delta$  such that for  $\zeta \in D_I^*$*

$$\begin{aligned} |\nabla_\zeta E_I f(\zeta)| &\leq C \|f\|_\beta [\text{dist}(\zeta, M) + \delta]^{-1/2-\beta}, \quad \text{if } \epsilon = 0 \\ |\nabla_\zeta E_I f(\zeta)| &\leq C \|f\|_\beta [\text{dist}(\zeta, M) + \delta]^{-1-\beta}, \quad \text{if } \epsilon = -1 \\ |\nabla_\zeta E_I f(\zeta)| &\leq C \|f\|_\beta [\text{dist}(\zeta, M) + \delta]^{-\beta}, \quad \text{if } \epsilon = 1 \end{aligned}$$

Using the classical Hardy-Littlewood lemma, we deduce from Lemma 2.2 and Lemma 2.3, the following estimates for the operators  $\tilde{C}_{I*}$  and  $X_\mathbb{C}^\zeta \tilde{C}_{I*}$

**Proposition 2.4.** *Let  $f \in \mathcal{B}_*^\beta(\Gamma_I)$ ,  $0 \leq \beta < 1$ , be a form with compact support in  $D$ , then*

$$\begin{aligned} \tilde{C}_{I*} f &\in \mathcal{C}_*^{1/2-\beta}(D_I^*), \quad \text{if } 0 \leq \beta < 1/2 \\ \tilde{C}_{I*} f &\in \mathcal{B}_*^{\beta-1/2}(D_I^*), \quad \text{if } 1/2 \leq \beta < 1 \end{aligned}$$

and

$$X_\mathbb{C}^\zeta \tilde{C}_{I*} f \in \mathcal{B}_*^\beta(D_I^*).$$

Let us recall Lemma 5.3 and 5.5 from [2]

**Lemma 2.5.** *There exists  $Y_1, \dots, Y_k$  tangential vector fields to  $M$  such that for all  $\zeta \in U_0$  and all  $1 \leq i, j \leq k$*

$$Y_i^\zeta \Phi_j(\zeta, \zeta) = \delta_{ij},$$

where  $\delta_{ij}$  denotes the Kronecker index.

If  $X^z$  is a vector field on  $D$  in the variable  $z$ , we denote by  $X^\zeta$  the same vector field in the variable  $\zeta$ .

**Lemma 2.6.** *Let us fix  $\delta > 0$ , if  $\hat{E}_\delta$  is the kernel of an operator of type  $(m, \epsilon, \delta)$ , there exists  $\eta > 0$  such that for  $(z, \zeta) \in \overline{D}_I \times \overline{D}_I^*$  with  $|z - \zeta| < \eta$  and  $z \neq \zeta$  we have*

$$X^z \hat{E}_\delta = -X^\zeta \hat{E}_\delta + \frac{(X^z + X^\zeta) \Phi_{I*}}{Y_\lambda^\zeta \Phi_{I*}} Y_\lambda^\zeta \hat{E}_\delta + \hat{G}_\delta, \quad (2.5)$$

where  $\hat{G}_\delta$  is a finite sum of kernels associated to operators of type  $(m, \epsilon, \delta)$ .

As regularity is a local problem, let us fix  $\zeta_0$  in  $D \cap M$  and choose a function  $\chi$  with compact support such that  $\chi(z) = 1$ , if  $|z - \zeta_0| \leq \eta/4$ , and  $\chi(z) = 0$ , if  $|z - \zeta_0| \geq \eta/2$ , where  $\eta$  is given by Lemma 2.6.

If  $f$  is a continuous form on  $D \cap M$  and  $f_I$  a continuous extension of  $f$  to  $\Gamma_I$ , we write

$$\tilde{C}_{I*}f(\zeta) = \tilde{C}_{I*}(\chi f)(\zeta) + \tilde{C}_{I*}((1 - \chi)f)(\zeta)$$

As the singularity of the kernel associated to the operator  $\tilde{C}_{I*}$  is concentrated on the diagonal, the regularity of the second term, for  $\zeta \in M$  with  $|\zeta - \zeta_0| \leq \eta/4$ , is directly given by the regularity of this kernel, which is of class  $\mathcal{C}^\infty$  in the first variable  $z$  and of class  $\mathcal{C}^{l-2}$  in the second variable  $\zeta$  when  $M$  is of class  $\mathcal{C}^l$ .

**Theorem 2.7.** *Assume  $M$  is of class  $\mathcal{C}^l$ . The operator  $\tilde{R}_M = \sum_{I \in \mathcal{I}'(k)} \text{sgn}(I) \tilde{C}_{I*}$  is a bounded linear operator from  $\mathcal{A}_*^{p+\alpha}(\overline{D} \cap M)$  into  $\mathcal{A}_*^{p+1+\alpha}(\overline{D} \cap M)$  for  $0 \leq p \leq 2l - 7$  and  $0 < \alpha < 1$ .*

*Proof.* Let us first consider the case  $p = 0$ . Let  $f \in \mathcal{A}_*^\alpha(\overline{D} \cap M)$ , i.e.  $f \in \mathcal{C}_*^{\alpha/2}(\overline{D} \cap M)$ , then there exists, for each  $I \in \mathcal{I}'(k)$ , a continuous form  $f_I$  on  $\overline{\Gamma}_I$  such that  $f_{I|M} = f$  and  $X_I f_I \in \mathcal{B}_*^{1-\alpha/2}(\Gamma_I)$ , for all vector fields  $X_I$  tangent to  $\Gamma_I$ . From the previous remark we have only to study  $\tilde{C}_{I*}(\chi f)(\zeta)$ . Let us denote by  $\tilde{C}_{I*}^\delta$  the operator  $\tilde{C}_{I*}$ , in which we have replaced  $\Phi$  by  $\Phi + \delta$ . From Lemma 2.6, we get that for all vector fields  $X_I$  tangent to  $\overline{\Gamma}_I \cup \overline{\Gamma}_I^*$ , we have  $X_I^\zeta \tilde{C}_{I*}^\delta \chi f = \tilde{C}_{I*}^\delta X_I^\zeta \chi f_I + E_I \chi f$ , where  $E_I$  is a sum of operators of type  $(m, \epsilon, \delta)$ ,  $\epsilon \geq 0$ . Let also  $X_{\mathbb{C}}$  be a complex tangential vector field to  $M$ .

We deduce from Lemma 2.2 that, for  $\zeta \in \Gamma_I^*$ ,

$$|X_I^\zeta \tilde{C}_{I*}^\delta(\chi f)(\zeta)| \leq C \|X_I(\chi f_I)\|_\beta [\text{dist}(\zeta, M) + \delta]^{(\alpha-1)/2}$$

and

$$|X_I^\zeta X_{\mathbb{C}} \tilde{C}_{I*}^\delta(\chi f)(\zeta)| \leq C' \|X_I f_I\|_\beta [\text{dist}(\zeta, M) + \delta]^{\alpha/2-1},$$

where  $C$  and  $C'$  are constants independent of  $\delta$ . Moreover  $X_I^\zeta \tilde{C}_{I*}^\delta f$  and  $X_I^\zeta X_{\mathbb{C}} \tilde{C}_{I*}^\delta f$  converge respectively to  $X_I^\zeta \tilde{C}_{I*} f$  and  $X_I^\zeta X_{\mathbb{C}} \tilde{C}_{I*} f$  uniformly on each compact subset of  $\Gamma_I^*$ , when  $\delta$  tends to zero. This implies that  $X_I^\zeta \tilde{C}_{I*} f \in \mathcal{B}_*^{(1-\alpha)/2}(\Gamma_I^*)$  and  $X_I^\zeta X_{\mathbb{C}} \tilde{C}_{I*} f \in \mathcal{B}_*^{1-\alpha/2}(\Gamma_I^*)$  and by the classical Hardy-Littlewood lemma that  $\tilde{C}_{I*} f \in \mathcal{C}_*^{(\alpha+1)/2}(\overline{D} \cap M)$  and  $X_{\mathbb{C}} \tilde{C}_{I*} f \in \mathcal{C}_*^{\alpha/2}(\overline{D} \cap M)$ , which means that  $\tilde{C}_{I*} f \in \mathcal{A}_*^{1+\alpha}(\overline{D} \cap M)$ . This ends the proof of this case by definition of the operator  $R_M$ .

Assume now that  $p = 1$ . Let  $f \in \mathcal{A}_*^{1+\alpha}(\overline{D} \cap M)$ , i.e.  $f \in \mathcal{C}_*^{(1+\alpha)/2}(\overline{D} \cap M)$  and  $X_{\mathbb{C}} f \in \mathcal{A}_*^\alpha(\overline{D} \cap M)$ , where  $X_{\mathbb{C}}$  is complex tangent to  $M$ .

If we proceed as in the case  $p = 0$ , but using Lemma 2.3 at the place of Lemma 2.2, we can prove that, for each  $I \in \mathcal{I}'(k)$  and each vector field  $X$  tangent to  $M$ ,  $\nabla_\zeta \tilde{C}_{I*}(\chi X^z f) \in \mathcal{B}_*^{1-\alpha/2}(D_I^*)$ , which implies by the classical Hardy-Littlewood lemma that  $\tilde{C}_{I*}(\chi X^z f) \in \mathcal{C}_*^{\alpha/2}(\overline{D} \cap M)$ .

Using the case  $p = 0$ , we get that if  $X_{\mathbb{C}} f \in \mathcal{A}_*^\alpha(\overline{D} \cap M)$ , then  $\tilde{R}_M \chi X_{\mathbb{C}} f \in \mathcal{A}_*^{1+\alpha}(\overline{D} \cap M)$ . Moreover it follows from the proof of Theorem 5.6 in [2] that if  $X_{\mathbb{C}} f$  is continuous then  $X_{\mathbb{C}} \tilde{R}_M(\chi f)$  and  $\tilde{R}_M \chi X_{\mathbb{C}} f$  have the same regularity, consequently  $X_{\mathbb{C}} \tilde{R}_M(\chi f) \in$

$\mathcal{A}_*^{1+\alpha}(\overline{D} \cap M)$ . By definition of the space  $\mathcal{A}_*^{2+\alpha}(\overline{D} \cap M)$ , we have then proved that  $\tilde{R}_M(\chi f) \in \mathcal{A}_*^{2+\alpha}(\overline{D} \cap M)$ .

Since by the proof of Theorem 5.6 in [2], if  $Xf$  is continuous for any vector field  $X$  tangent to  $M$ , then  $X\tilde{R}_M f$  and  $\tilde{R}_M \chi Xf$  have the same regularity, the theorem follows for  $p \geq 2$ , by a simple induction, from the definition of the spaces  $\mathcal{A}_*^{p+\alpha}(\overline{D} \cap M)$ .  $\square$

*Remark 2.8.* Note that if we exchange the role played by the variables  $z$  and  $\zeta$  in the operator  $\tilde{R}_M$ , then Theorem 2.7 is valid for  $0 \leq p \leq 2l - 6$

Theorem 2.7 implies better regularity properties for the operators involve in Proposition 1.1 and Theorem 1.2.

**Corollary 2.9.** *Under the hypotheses of Proposition 1.1 and Theorem 1.2, the operators  $T_r$ ,  $1 \leq r \leq q$  and  $n - k - q + 1 \leq r \leq n - k$ , in Proposition 1.1 are bounded linear operators from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M)$  into  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(\overline{U})$  and the operators  $A_r$ ,  $1 \leq r \leq q$  and  $n - k - q + 1 \leq r \leq n - k$ , in Theorem 1.2 are bounded linear operators from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  into  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(M, E)$ .*

*Proof.* Since the operators  $T_r$  from Proposition 1.1 are defined as the restrictions to  $\mathcal{C}_{n,r}^0(M)$  of the operators  $\tilde{R}_M$ , if  $1 \leq r \leq q - 1$ , and as the restrictions to  $\mathcal{C}_{n,r}^0(M)$  of the operators  $\tilde{R}_M$  after exchanging the variables  $z$  and  $\zeta$  in the kernels, if  $n - k - q + 1 \leq r \leq n - k$ , they are in fact bounded from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M)$  into  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(\overline{U})$ .

Assume  $M$  is a compact  $q$ -concave generic CR submanifold of a complex manifold  $X$  and  $E$  is an holomorphic vector bundle on  $X$ . Let us go back to the construction of the global operators  $A_r$  (cf. [7] and [3]). First by globalizing the homotopy formula 1.2 by mean of a partition of unity, one get new linear operators  $\tilde{T}_r$  bounded from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  into  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(M, E)$  and  $K_r$  bounded from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  into  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  such that

$$f + K_r f = \begin{cases} \tilde{T}_1 \bar{\partial}_b f & \text{if } r = 0, \\ \bar{\partial}_b \tilde{T}_r f + \tilde{T}_{r+1} \bar{\partial}_b f & \text{if } 1 \leq r \leq r - 1 \text{ or } n - k - q + 1 \leq r \leq n - k. \end{cases} \quad (2.6)$$

From functional analysis arguments we get linear operators  $T'_r$ ,  $1 \leq r \leq q$  and  $n - k - q + 1 \leq r \leq n - k$ , bounded from  $\mathcal{C}_{n,r}^0(M, E)$  into  $\mathcal{C}_{n,r-1}^\infty(M, E)$  and  $K'_r$ ,  $0 \leq r \leq q - 1$  and  $n - k - q + 1 \leq r \leq n - k$ , bounded from  $\mathcal{C}_{n,r}^0(M, E)$  into  $\mathcal{C}_{n,r}^\infty(M, E)$  such that if

$$N_r = I + K_r + K'_r \quad (2.7)$$

then

$$\mathcal{C}_{n,r}^0(M, E) = \text{Im} N_r \oplus \text{Ker} N_r, \quad \text{Ker} N_r \subset \mathcal{C}_{n,r}^\infty(M, E).$$

We set  $\hat{N}_r = N_r + P_r = I + R_r$ , where  $P_r$  denotes the linear projection with  $\text{Im} P_r = \text{Ker} N_r$  and  $\text{Ker} P_r = \text{Im} N_r$ . The operator  $R_r$  is continuous from  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  into  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$  and hence compact as an endomorphism of  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$ . Consequently  $\hat{N}_r|_{\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)}$  is a Fredholm endomorphism with index 0 and as  $\text{Ker} \hat{N}_r = \{0\}$ , it is an isomorphism. Moreover  $\hat{N}_r^{-1} = I - R_r \hat{N}_r^{-1}$ , this implies that  $\hat{N}_r^{-1}|_{\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)}$  is a continuous endomorphism of  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$ . The operators  $A_r$  are then defined by

$$A_r = \begin{cases} \hat{N}_{r-1}^{-1}(T_r + T'_r), & 1 \leq r \leq q \\ (T_r + T'_r)\hat{N}_r^{-1}, & n - k - q + 1 \leq r \leq n - k, \end{cases} \quad (2.8)$$

and hence continuous from  $\mathcal{A}_{n,r}^{p+\alpha}(M, E)$  into  $\mathcal{A}_{n,r}^{p+1+\alpha}(M, E)$ .  $\square$

### 3 Second anisotropic estimates

When they studied the tangential Cauchy-Riemann complex on the Heisenberg group and more generally on strictly pseudoconvex CR manifolds, Folland and Stein have introduced some anisotropic Hölder spaces.

Let  $M$  be a generic CR manifold of class  $\mathcal{C}^3$  and of real codimension  $k$  in a complex manifold  $X$  of complex dimension  $n$  and  $D$  be a relatively compact domain in  $X$ . Let  $X_1, \dots, X_{2n-2k}$  be a real basis of  $HM$ . A  $\mathcal{C}^1$  curve  $\gamma : [0, r] \rightarrow M$  is called admissible if for every  $t \in [0, r]$ ,

$$\frac{d\gamma}{dt}(t) = \sum_{j=1}^{2n-2k} c_j(t) X_j(\gamma(t))$$

where  $\sum |c_j(t)|^2 \leq 1$ .

The Folland-Stein anisotropic Hölder spaces  $\Gamma^{p+\alpha}(\overline{D} \cap M)$  are defined in the following way:

-  $\Gamma^\alpha(\overline{D} \cap M)$ ,  $0 < \alpha < 1$ , is the set of continuous functions in  $\overline{D} \cap M$  such that if for every  $x_0 \in \overline{D} \cap M$

$$\sup_{\gamma(\cdot)} \frac{|f(\gamma(t)) - f(x_0)|}{|t|^\alpha} < \infty$$

for any admissible complex tangent curve  $\gamma$  through  $x_0$ .

-  $\Gamma^{p+\alpha}(\overline{D} \cap M)$ ,  $p \geq 1$ ,  $0 < \alpha < 1$ , is the set of continuous functions in  $M$  such that  $X_{\mathbb{C}} f \in \Gamma^{p-1+\alpha}(\overline{D} \cap M)$ , for all complex tangent vector fields  $X_{\mathbb{C}}$  to  $M$ .

If  $M$  is  $q$ -concave,  $q \geq 1$ , the complex tangent vector fields and their first Lie brackets generates all the vector fields in a neighborhood of each point. Associated with the vector fields  $X_1, \dots, X_{2n-2k}$  we define a distance function  $\text{dist}(x, y)$  for any  $x, y \in M$  to be the infimum of the set of all  $r$  for which there exists an admissible curve with  $\gamma(0) = x$  and  $\gamma(r) = y$ . This distance function defines a family of nonisotropic balls  $B_r(x_0) = \{x \in M \mid \text{dist}(x_0, x) < r\}$  for each point  $x_0 \in M$ . Let  $T_1, \dots, T_k$  be  $k$  real vector fields such that  $X_1, \dots, X_{2n-2k}, T_1, \dots, T_k$  span the whole tangent space in a neighborhood of  $x_0$ . Then for  $r$  sufficiently small, the ball  $B_r(x_0)$  has length comparable to  $r$  in the direction of  $X_1, \dots, X_{2n-2k}$  and length comparable to  $r^2$  in the direction of  $T_1, \dots, T_k$  (cf. [8]).

A function  $u$  belongs to  $\Gamma^\alpha(\overline{D} \cap M)$  if  $u$  is continuous on  $\overline{D} \cap M$  and

$$|u(x) - u(y)| \leq C(\text{dist}(x, y))^\alpha \quad \text{for every } x, y \in \overline{D} \cap M$$

The Folland-Stein anisotropic Hölder space of forms  $\Gamma_*^{p+\alpha}(\overline{D} \cap M)$ ,  $p \geq 0$ ,  $0 < \alpha < 1$ , is then the space of continuous forms on  $\overline{D} \cap M$ , whose coefficients are in  $\Gamma^{p+\alpha}(\overline{D} \cap M)$ .

We will now prove sharp estimates for the operators  $\tilde{C}_{I*}$ ,  $I \in \mathcal{I}'(k)$ , defined in Section 2 with respect to these spaces, for this we may assume that  $M$  is embedded in  $\mathbb{C}^n$ .

**Proposition 3.1.** *Assume  $M$  is of class  $\mathcal{C}^\infty$  then the operators  $\tilde{C}_{I*}$ ,  $I \in \mathcal{I}'(k)$ , are continuous from  $\Gamma_*^{p+\alpha}(\overline{D} \cap M)$  into  $\Gamma_*^{p+1+\alpha}(\overline{D} \cap M)$ ,  $p \geq 0$ .*

*Remark 3.2.* Note that in fact to get Proposition 3.1 for some  $p$  the manifold  $M$  needs only to be of class  $\mathcal{C}^{p+4}$  and to be of class  $\mathcal{C}^{p+2}$  if we exchange the role of  $z$  and  $\zeta$  in the kernel  $C_{I*}$ .

*Proof.* For  $f \in \mathcal{C}_{n,r}(\overline{D})$ ,  $0 \leq r \leq n - k$ , we set

$$F(\zeta) = \tilde{C}_{I*}f(\zeta) = \int_{z \in D \cap M} f(z) \wedge C_{I*}(z, \zeta) = \int_{(z, \lambda) \in D \cap M \times \Delta_{I*}} f(z) \wedge K_{I*}(z, \zeta, \lambda).$$

Without loss of generality we can assume that  $f = \tilde{f}\theta$  with  $\tilde{f} \in \mathcal{C}_{0,0}(\overline{D})$  and  $\theta \in \mathcal{C}_{n,r}^\infty(\overline{D})$ . We set  $\psi(\zeta) = \tilde{C}_{I*}\theta(\zeta)$ , the differential form  $\psi$  is then of class  $\mathcal{C}^\infty$  (cf. Section 5 in [2]) and for  $\zeta_1$  and  $\zeta_2$  in  $\overline{D}$

$$F(\zeta_1) - F(\zeta_2) = \int_{z \in D \cap M} (\tilde{f}(z) - \tilde{f}(\zeta_1))\theta(z) \wedge (C_{I*}(z, \zeta_1) - C_{I*}(z, \zeta_2) + \tilde{f}(\zeta_1)(\psi(\zeta_1) - \psi(\zeta_2))).$$

We first consider the case where  $p = 0$ . Let  $\gamma$  be an admissible curve with  $\gamma(0) = \zeta_1$  and  $\gamma(r) = \zeta_2$  we have to estimate  $\frac{d(F \circ \gamma)}{dt}(0) - \frac{d(F \circ \gamma)}{dt}(r)$  when  $\tilde{f} \in \Gamma^\alpha(\overline{D} \cap M)$ . We have

$$\begin{aligned} \frac{d(F \circ \gamma)}{dt}(0) - \frac{d(F \circ \gamma)}{dt}(r) &= \int_{z \in D \cap M} (\tilde{f}(z) - \tilde{f}(\gamma(0)))\theta(z) \wedge \left( \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(0) - \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right) \\ &\quad + \tilde{f}(\gamma(0)) \left( \frac{d\psi \circ \gamma}{dt}(0) - \frac{d\psi \circ \gamma}{dt}(r) \right). \end{aligned}$$

We set  $D_1 = \{z \in \overline{D} \cap M \mid |z - \zeta_1| \leq 2|\zeta_1 - \zeta_2|\}$  and  $D_2 = \{z \in \overline{D} \cap M \mid |z - \zeta_1| \geq 2|\zeta_1 - \zeta_2|\}$  then  $\overline{D} \cap M = D_1 \cup D_2$ . We then have

$$\left| \frac{d(F \circ \gamma)}{dt}(0) - \frac{d(F \circ \gamma)}{dt}(r) \right| \leq J_1 + J_2 + |\tilde{f}(\gamma(0))| \left( \frac{d\psi \circ \gamma}{dt}(0) - \frac{d\psi \circ \gamma}{dt}(r) \right)|$$

where

$$\begin{aligned} J_1 &= \left| \int_{z \in D_1} (\tilde{f}(z) - \tilde{f}(\gamma(0)))\theta(z) \wedge \left( \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(0) - \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right) \right| \\ &\leq \int_{z \in D_1} |\tilde{f}(z) - \tilde{f}(\gamma(0))| |\theta(z)| \wedge \left| \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(0) \right| \\ &\quad + \int_{z \in D_1} |\tilde{f}(z) - \tilde{f}(\gamma(r))| |\theta(z)| \wedge \left| \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right| \\ &\quad + |\tilde{f}(\gamma(0)) - \tilde{f}(\gamma(r))| \int_{z \in D_1} |\theta(z)| \wedge \left| \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right|. \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} J_2 &= \left| \int_{z \in D_2} (\tilde{f}(z) - \tilde{f}(\gamma(0)))\theta(z) \wedge \left( \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(0) - \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right) \right| \\ &\leq \int_{z \in D_2} |\tilde{f}(z) - \tilde{f}(\gamma(0))| |\theta(z)| \wedge \left| \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(0) - \frac{dC_{I*}(z, \gamma(\cdot))}{dt}(r) \right|. \end{aligned}$$

Let us first consider  $J_1$ . The last term in (3.1) is clearly controlled by  $C\|\tilde{f}\|_{\Gamma^\alpha} r^\alpha$  and both other integral terms in (3.1) are of the same type.

Note that since  $\gamma$  is a complex tangential curve  $\frac{dC_{I^*}(z, \gamma(\cdot))}{dt}(s)$ ,  $s = 0, r$ , involves only complex tangential derivatives of the part of degree  $k$  in  $\lambda$  of the kernel  $K_{I^*}(z, \zeta, \lambda)$ . As  $X_C^\zeta \phi(z, \zeta, \lambda) = O_1$  the coefficients of these derivatives are sums of terms of type  $\frac{O_{k-m}}{\phi^n}$ ,  $0 \leq m \leq k$ . Moreover the first order terms in  $\Phi(z, \zeta, \lambda)$  are equivalent to the distance of  $z$  and  $\zeta$  in the real tangential directions transverse to the complex tangent space (i.e. in the directions span by  $T_1, \dots, T_k$  for a good choice of  $T_1, \dots, T_k$ ). Therefore  $[\text{dist}(z, \zeta)]^2 = O(|\Phi(z, \zeta, \lambda)|)$  and if  $\tilde{f} \in \Gamma^\alpha(\overline{D} \cap M)$  we get for  $s = 0, r$

$$|\tilde{f}(z) - \tilde{f}(\gamma(s))| = O(|\Phi(z, \gamma(s), \lambda)|^{\alpha/2}).$$

The same calculations as in Section 5.1 of [2] lead to the estimate

$$J_1 \leq C\|\tilde{f}\|_{\Gamma^\alpha} r^\alpha.$$

To estimate  $J_2$ , it follows from Section 5.1 of [2] that the main point is to control the difference

$$\frac{1}{\Phi(z, \gamma(0), \lambda)} - \frac{1}{\Phi(z, \gamma(r), \lambda)}.$$

Since  $[d_\zeta \Phi(z, \gamma(t), \lambda) \cdot \gamma'(t)]|_{t=r} = O_1$  we have

$$\left| \frac{1}{\Phi(z, \gamma(0), \lambda)} - \frac{1}{\Phi(z, \gamma(r), \lambda)} \right| \leq C \sum_{p=0}^{n-1} \frac{|\gamma(0) - \gamma(r)| O_1 + |\gamma(0) - \gamma(r)|^2 O_0}{|\Phi^{n-p}(z, \gamma(0), \lambda)| |\Phi^{p+1}(z, \gamma(r), \lambda)|}.$$

As for  $J_1$  following the estimations in Section 5.1 of [2] we get

$$J_2 \leq C\|\tilde{f}\|_{\Gamma^\alpha} r^\alpha$$

and finally

$$\left| \frac{d(F \circ \gamma)}{dt}(0) - \frac{d(F \circ \gamma)}{dt}(r) \right| \leq C\|\tilde{f}\|_{\Gamma^\alpha} r^\alpha$$

since  $\tilde{f}$  is bounded and  $\psi$  is of class  $\mathcal{C}^1$ .

Using Lemma 2.6 we can derive the case  $p \geq 1$  from the case  $p = 0$  in the same way as in the previous section and in Section 5.2 in [2].  $\square$

It follows from Proposition 3.1 and from the definition of the operators  $T_r$  and  $A_r$ ,  $1 \leq r \leq q$  and  $n - k - q + 1 \leq r \leq n - k$ , from Section 1 that Corollary 2.9 still holds if we replace the spaces  $\mathcal{A}_{n,r}^{p+\alpha}(M)$  and  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(\overline{U})$  by the spaces  $\Gamma_{n,r}^{p+\alpha}(M)$  and  $\Gamma_{n,r-1}^{p+1+\alpha}(\overline{U})$  respectively and the spaces  $\mathcal{A}_{n,r}^{p+\alpha}(M, E)$  and  $\mathcal{A}_{n,r-1}^{p+1+\alpha}(M, E)$  by the spaces  $\Gamma_{n,r}^{p+\alpha}(M, E)$  and  $\Gamma_{n,r-1}^{p+1+\alpha}(M, E)$  respectively.

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